

# Evolution of Cosmological Perturbations in the Long Wavelength Limit

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## Abstract

The relation between the long wavelength limit of solutions to the cosmological perturbation equations and the perturbations of solutions to the exactly homogeneous background equations is investigated for scalar perturbations on spatially flat cosmological models. It is shown that a homogeneous perturbation coincides with the long wavelength limit of some inhomogeneous perturbation only when the former satisfies an additional condition corresponding to the momentum constraint if the matter consists only of scalar fields. In contrast, no such constraint appears if the fundamental variables describing the matter contain a vector field as in the case of a fluid. Further, as a byproduct of this general analysis, it is shown that there exist two universal exact solutions to the perturbation equations in the long wavelength limit, which are expressed only in terms of the background quantities. They represent adiabatic growing and decaying modes, and correspond to the well-known exact solutions for perfect fluid systems and scalar field systems.

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# §1 Introduction

In the current standard scenario based on the gravitational instability theory, the present large scale structures of the universe are formed through the following four stages: production of seed fluctuations in the early universe, their linear evolution on superhorizon scales, the subsequent linear modulation after they enter the horizon, and the final non-linear evolutions. In the inflationary universe models the seed fluctuations in the first stage are produced from quantum fluctuations on the Hubble horizon scales during inflation, and we now have universal formulae to determine their nature such as the amplitudes and the spectrum for a wide variety of inflation models. We can now also easily handle the evolution during the third stage because the matter content of the universe during this stage is rather simple and restricted. Of course the actual behavior of perturbations on subhorizon scales are quite complicated even in the linear stage, and we need numerical computations to determine their details.

On the other hand universal formulae to determine the evolution of perturbations during the second stage have not been established yet, although it is generally believed that the so-called Bardeen parameter is conserved with good accuracy during this stage and this conservation law essentially determines the amplitudes and the spectrum of perturbations when they reenter the horizon, from those at the first stage[1]. Of course the conservation of the Bardeen parameter during the Friedmann stage is well established including the case in which the equation of state of cosmic matter changes slowly [2, 3]. The simplicity comes from that fact that the evolution of perturbations on superhorizon scales are determined with good accuracy by that on the long wavelength limit [4, 5, 6]. In this limit the evolution of adiabatic modes is determined by a set of simple 1st-order ordinary differential equations for two gauge-invariant variables, from which follows the exact conservation of the Bardeen parameter [2, 3]. However, for some important situations, such as the reheating stage of inflation models, this simple analysis does not work because the adiabatic mode may produce entropy modes which feed back to the behavior of the adiabatic modes[7]. In these situations the knowledge on the evolution of perturbations of all the components of matter in the long wavelength limit is required to determine the behavior of the Bardeen parameter.

In this connection Nambu and Taruya recently wrote an interesting paper [8] in which they stated that solutions to the gauge-invariant perturbation equations for a multi-component scalar field system on expanding universe in the long wavelength limit are obtained as derivatives of exactly homogeneous solutions to the Einstein equations with respect to the solution parameters. If this result is correct and all the solutions to the perturbation equations can be obtained in this way, it provides a very powerful tool to analyze the behavior of perturbations during inflation and reheating including the problem of the conservation of the Bardeen parameter. However, no proof nor explanation on their statement was given in their paper. Further, direct calculation shows that the perturbation equations in the long wavelength limit do not coincide with those for exactly homogeneous perturbations.

In order to see whether Nambu and Taruya's statement is true or not as well as to obtain a deeper understanding on the behavior of perturbations on superhorizon scale perturbations, in the present paper, we investigate the relation between the solutions to the perturbation equations in long wavelength limit and the exactly homogeneous solutions

to the Einstein equations for multi-component systems on spatially flat Robertson-Walker universe. Our arguments are quite general except for the assumptions that the fundamental variables describing matter are scalar fields and/or vector fields, and that the amplitude of anisotropic stress perturbations vanishes rapidly enough in the long wavelength limit. We will clarify general conditions under which the perturbation solutions in the long wavelength limit coincide with some exactly homogeneous perturbations to the homogeneous solutions to the Einstein equations. In particular, we will show that for the multi-component scalar field system not all the exactly homogeneous perturbations directly correspond to perturbation solutions in the long wavelength limit. Further, as a byproduct of our analysis, we will show that the perturbation equations in the long wavelength limit have two universal adiabatic solutions which can be expressed explicitly as time integrals of known background quantities.

The paper is organized as follows. First in the next section we give basic definitions of perturbation variables used in the present paper and the perturbation equations for them. By inspecting the dependence of these quantities and equations on the wave number  $k$  of perturbations, in §3 we derive the conditions on the exactly homogeneous perturbations to coincide with the  $k \rightarrow 0$  limit of some solutions to the perturbation equations with  $k \neq 0$ . In §4, as a special application of the argument in §3, we show the existence of two universal adiabatic modes in the  $k \rightarrow 0$  limit. Then in §5 and §6 we specify our arguments to a multi-component scalar field system and a multi-component perfect fluid system, respectively, and clarify the relation between the perturbations in the  $k \rightarrow 0$  limit and those with  $k = 0$ . Section 7 is devoted to summary and discussions.

## §2 Perturbation Equations

In this section we recapitulate the definitions of basic perturbation variables and their equations in the framework of the gauge-invariant perturbation theory. We adopt the notations used in the review article by Kodama and Sasaki[2].

We only consider perturbations on a spatially flat( $K = 0$ ) Robertson-Walker universe throughout the paper. Hence the background metric is given by

$$ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2, \quad (2.1)$$

and its perturbation by

$$d\tilde{s}^2 = -(1 + 2AY)dt^2 - 2aBY_j dt dx^j + a^2 [(1 + 2H_L Y)\delta_{jk} + 2H_T Y_{jk}] dx^j dx^k, \quad (2.2)$$

where  $Y$ ,  $Y_j$  and  $Y_{jk}$  are harmonic scalar, vector and tensor for a scalar perturbation with wave vector  $\mathbf{k}$  on flat 3-space:

$$Y := e^{i\mathbf{k} \cdot \mathbf{x}}, \quad Y_j := -i \frac{k_j}{k} Y, \quad Y_{jk} := \left( \frac{1}{3} \delta_{jk} - \frac{k_j k_k}{k^2} \right) Y. \quad (2.3)$$

Under the infinitesimal gauge transformation

$$\delta t = TY, \quad \delta x^j = LY^j, \quad (2.4)$$

the metric perturbation variables in Eq.(2.2) transform as

$$\bar{A} = A - \dot{T}, \quad (2.5)$$

$$\bar{B} = B + a\dot{L} + \frac{k}{a}T, \quad (2.6)$$

$$\bar{H}_L = H_L - \frac{k}{3}L - HT, \quad (2.7)$$

$$\bar{H}_T = H_T + kL \quad (2.8)$$

where the overdot represents the derivative with respect to the proper time  $t$  and  $H$  is the cosmic expansion rate  $\dot{a}/a$ . From this it follows that the spatial curvature perturbation  $\mathcal{R}$  and the shear  $\sigma_g$  defined by

$$\mathcal{R} := H_L + \frac{1}{3}H_T, \quad (2.9)$$

$$\sigma_g := \frac{a}{k}\dot{H}_T - B \quad (2.10)$$

transform as

$$\bar{\mathcal{R}} = \mathcal{R} - HT, \quad \bar{\sigma}_g = \sigma_g - \frac{k}{a}T. \quad (2.11)$$

Hence we obtain the following two independent gauge-invariant combinations:

$$\mathcal{A} := A - (\mathcal{R}/H)', \quad (2.12)$$

$$\Phi := \mathcal{R} - \frac{aH}{k}\sigma_g. \quad (2.13)$$

All the other gauge-invariant combinations of metric perturbation variables are written as linear combinations of  $\mathcal{A}$  and  $\Phi$ .

Here note that for the exactly homogeneous perturbations corresponding to  $k = 0$ , the vector-like and the tensor-like perturbation variables should vanish. Hence for the metric perturbation,  $B = H_T = 0$ . Correspondingly, the gauge freedom for them is just the time reparametrization  $T$ , and  $L$  should vanish in the gauge transformation. Therefore there exists only one gauge-invariant combination for the exactly homogeneous metric perturbations, which coincides with  $\mathcal{A}$ . The variable  $\Phi$  has no counterpart for them. This point is very important in the argument on the correspondence between the perturbations in the  $k \rightarrow 0$  limit and the exactly homogeneous ones.

Next we consider the matter perturbations. In this paper we assume that the fundamental variables describing matter are scalar and vector quantities. This assumption is satisfied in most of the realistic applications. Let us denote the corresponding perturbation variables by  $\chi_I$  and  $v_P$  where  $I$  and  $P$  are indices labeling components. Then they transform under the gauge transformation as

$$\bar{\chi}_I = \chi_I - \dot{S}_I T, \quad \bar{v}_P = v_P + a\dot{L}, \quad (2.14)$$

where  $S_I$  is the background quantity corresponding to the perturbation  $\chi_I$ . Hence the corresponding gauge-invariant variables are given by

$$X_I := \chi_I - \frac{\dot{S}_I}{H}\mathcal{R}, \quad (2.15)$$

$$V_P := v_P - \frac{a}{k}\dot{H}_T. \quad (2.16)$$

In the exactly homogeneous system the matter is described only by the scalar quantities  $S_I$ . Hence its perturbation gives only the scalar-type gauge-invariant variables  $X_I$  as in the case of the metric perturbation.

The Einstein equations relate these matter variables with the metric variables through the energy-momentum tensor. For scalar perturbations its generic form is written as

$$\tilde{T}_0^0 = -(\rho + \delta\rho Y), \quad (2.17)$$

$$\tilde{T}_j^0 = a(\rho + p)(v - B)Y_j, \quad (2.18)$$

$$\tilde{T}_k^j = (p\delta_k^j + \delta p\delta_k^j + \Pi Y_k^j), \quad (2.19)$$

where  $\rho$  and  $p$  are the background values of the energy density and the pressure, respectively, and follow the equations,

$$H^2 = \frac{\kappa^2}{3}\rho, \quad (2.20)$$

$$\dot{\rho} = -3(\rho + p)H. \quad (2.21)$$

In the present paper we only consider the case in which the anisotropic stress perturbation  $\Pi$ , which is gauge-invariant by itself, vanishes faster than  $k^2$  in the  $k \rightarrow 0$  limit.

Applying the general argument on the matter perturbation above, we can construct from the density perturbation  $\delta\rho$ , the velocity perturbation  $v$  and the isotropic stress perturbation  $\delta p$  the following three gauge-invariant combinations:

$$\rho\Delta_g := \delta\rho + 3(\rho + p)\mathcal{R}, \quad (2.22)$$

$$V := v - \frac{a}{k}\dot{H}_T, \quad (2.23)$$

$$p\Gamma := \delta p - c_s^2\delta\rho, \quad (2.24)$$

where  $c_s^2 := \dot{p}/\dot{\rho}$  is the square of the sound velocity.

Now in terms of these gauge-invariant variables the perturbation of the Einstein equations are written as

$$\tilde{G}_0^0 = \kappa^2\tilde{T}_0^0 : \quad \mathcal{A} + \frac{1}{2}\Delta_g = \frac{k^2}{3a^2H^2}\Phi, \quad (2.25)$$

$$\tilde{G}_j^0 = \kappa^2\tilde{T}_j^0 : \quad k \left[ \mathcal{A} + \frac{3}{2}(1 + w)Z \right] = 0, \quad (2.26)$$

$$\tilde{G}_j^j = \kappa^2\tilde{T}_j^j : \quad H\dot{\mathcal{A}} + (2\dot{H} + 3H^2)\mathcal{A} = \frac{\kappa^2}{2}(p\Gamma + c_s^2\rho\Delta_g) - \frac{\kappa^2}{3}\Pi, \quad (2.27)$$

$$\tilde{G}_k^j - \tilde{G}_l^l\delta_k^j = \kappa^2(\tilde{T}_k^j - \tilde{T}_l^l\delta_k^j) : \quad k^2 \left[ \mathcal{A} + \frac{1}{a} \left( \frac{a}{H}\Phi \right) \right] = -\kappa^2 a^2 \Pi, \quad (2.28)$$

where  $w = p/\rho$  and  $Z$  is the Bardeen parameter defined by

$$Z := \Phi - \frac{aH}{k}V = \mathcal{R} - \frac{aH}{k}(v - B). \quad (2.29)$$

If we eliminate  $\Delta_g$  from Eq.(2.27) using Eq.(2.25), we obtain

$$\left( \frac{\mathcal{A}}{1 + w} \right)' = \frac{c_s^2}{1 + w} \frac{1}{H} \frac{k^2}{a^2} \Phi + \frac{H}{\rho + p} \left( \frac{3}{2}p\Gamma - \Pi \right). \quad (2.30)$$

Note that we have not multiplied any power of  $k$  in deriving these equations from the Einstein equations. Hence putting  $k = 0$  in these equations yields the equations for the exactly homogeneous perturbations if one takes account of the fact that the terms multiplied by inverse powers of  $k$  vanishes identically for these perturbations as mentioned above.

In general these equations must be supplemented by the expressions for  $\Delta_g$ ,  $Z$ ,  $\Gamma$  and  $\Pi$  in terms of the fundamental gauge-invariant matter variables  $X_I$  and  $V_P$  and their evolution equations. We do not write them explicitly here because the details of these equations are not necessary until we specialize the general arguments to specific models.

### §3 $k = 0$ vs $k \rightarrow 0$ limit

In this section we clarify under what conditions the  $k \rightarrow 0$  limit of solutions to Eqs.(2.25)-(2.28) coincide with some solutions to the corresponding equations for  $k = 0$ . For that purpose first note that the gauge-invariant variables introduced in the previous section are classified into two groups.

The first one consists of the gauge-invariant variables whose expressions in terms of the gauge-variant perturbation variables do not contain inverse powers of  $k$ .  $\mathcal{A}$ ,  $\Delta_g$ ,  $\Gamma$  and  $X_I$  belong to this group. These variables always have well-defined finite  $k \rightarrow 0$  limits which coincide with the corresponding gauge-invariant variables for the exactly homogeneous perturbations. Of course in taking  $k \rightarrow 0$  limit the conditions

$$B, H_T, v_P, v, \Pi/k^2 \rightarrow 0 \quad (3.1)$$

should be taken into account.

On the other hand the second group consists of the gauge-invariant variable which contain inverse powers of  $k$  when expressed in terms of gauge-variant variables.  $\Phi$ ,  $Z$ (or  $V$ ) and  $V_P$  belong to this group. These variables have no counterpart for the exactly homogeneous perturbations, and their  $k \rightarrow 0$  limits depend on how fast the quantities in Eq.(3.1) vanish in the limit.

With these points in mind, let us first consider the  $k \rightarrow 0$  limit of the perturbation of the equations for the fundamental matter variables. When expressed in terms of the original gauge-variant perturbation variables  $\chi_I$ ,  $v_P$ ,  $A$ ,  $B$ ,  $H_L$  and  $H_T$ , they do not contain inverse powers of  $k$ . We can write them in terms of gauge-invariant variables in the following way. First eliminate  $\chi_I$  and  $A$  by replacing them by  $X_I$ ,  $\mathcal{A}$  and  $\mathcal{R}$ . Second eliminate  $v_P$  using  $V_{PQ} = v_P - v_Q$  and  $Z$ . These procedures do not produce terms with inverse powers of  $k$ , and the coefficient of  $Z$  contains a positive power of  $k$ . Now since all the gauge-variant matter variables and  $A$  are eliminated, the remaining terms should be proportional to  $\Phi$  (and its time derivatives). This implies that in the final expressions  $Z$  and  $\Phi$  appears only in the forms  $k^m Z (m \geq 1)$  and  $k^n \Phi (n \geq 2)$ . Further the other terms, which are proportional to  $\mathcal{A}$ ,  $X_I$  or  $V_{PQ}$ , contain no inverse power of  $k$ . Therefore, taking into account the condition (3.1), the  $k \rightarrow 0$  limit of a solution to the perturbation equations satisfies the corresponding equations for  $k = 0$  if and only if

$$k^2 \Phi \rightarrow 0, \quad (3.2)$$

$$kZ \rightarrow 0, \quad (3.3)$$

$$V_{PQ} \rightarrow 0. \quad (3.4)$$

Next let us examine the Einstein equations. Among the four equations it is easy to see that Eq.(2.25) and Eq.(2.27) reduce to the perturbation of Eq.(2.20) and Eq.(2.21) in the  $k \rightarrow 0$  limit under the condition  $k^2\Phi \rightarrow 0$ . On the other hand Eq.(2.26) and Eq.(2.28) become trivial for the exactly homogeneous perturbations. However, for  $k \neq 0$ , these equations yield non-trivial perturbation equations

$$\mathcal{A} + \frac{3}{2}(1+w)Z = 0, \quad (3.5)$$

$$\mathcal{A} + \frac{1}{a} \left( \frac{a}{H} \Phi \right)' = 0, \quad (3.6)$$

where we have put  $\Pi = 0$  because it does not affect the arguments on the  $k \rightarrow 0$  limit under the assumption  $\Pi/k^2 \rightarrow 0$  adopted in the present paper.

The second equation of these can be solved in terms of  $\Phi$  as

$$\Phi = \frac{H}{a} \left( C - \int_{t_0} a(t) \mathcal{A}(t) dt \right), \quad (3.7)$$

where  $C$  is an integration constant and  $t_0$  is an initial time. This equation can be regarded as an equation to determine the  $k \rightarrow 0$  limit of  $\Phi$  from a solution for  $\mathcal{A}$  to the perturbation equations with  $k = 0$ . In this viewpoint the consistency condition (3.2) is simply replaced by the condition on the  $k$ -dependence of  $C$ ,

$$k^2 C(k) \rightarrow 0. \quad (3.8)$$

On the other hand the condition (3.3) is always satisfied under Eq.(3.5). Further the condition (3.4) just select a subset of solutions to the perturbation equations with  $k \neq 0$ , and does not give any restriction on the exactly homogeneous perturbations. Therefore starting from any solution to the perturbation equations with  $k = 0$ , one can always construct the gauge-invariant quantities representing the  $k \rightarrow 0$  limit of a solution to the perturbation equations with  $k \neq 0$  by supplementing  $\mathcal{A}$  and  $X_I$  for the exactly homogeneous perturbation with  $\Phi$  determined by Eq.(3.7), provided that Eq.(3.5) do not yield any additional constraint on the seed exactly homogeneous perturbation.

In the case in which the fundamental variables describing matter contains a dynamical vector field, the  $k \rightarrow 0$  limit of  $Z$  depends on the value  $\lim_{k \rightarrow 0} (v - B)/k$  which cannot be determined from the information of the exactly homogeneous perturbations. Hence Eq.(3.5) can be simply regarded as the equation to determine the  $k \rightarrow 0$  limit of  $Z$ .

In contrast, in the case in which the matter is described only by scalar fields,  $v - B$  in  $\tilde{T}_j^0$  should be written as a combination of the spatial derivatives of  $\chi_I$ . Hence it must be proportional to a positive power of  $k$ , and  $Z$  has a well-defined  $k \rightarrow 0$  limit which is written only in terms of  $X_I$ . Thus the condition Eq.(3.5) yields a restriction on the seed exactly homogeneous perturbation in order for it to be a  $k \rightarrow 0$  limit of some solution to the perturbation equations with  $k \neq 0$ .

## §4 Universal Adiabatic Solutions

Let  $a(t)$  and  $S_I(t)$  be an exactly homogeneous solution describing the background. Then, since the scale factor  $a(t)$  comes into the evolution equations only through  $H = \dot{a}/a$ ,  $(1 + \lambda)a(t)$  and  $S_I(t)$  also gives an exactly homogeneous solution to the Einstein equation where  $\lambda$  is a constant. If we regard this solution as an exactly homogeneous perturbation, all the corresponding gauge-variant perturbation variables for matter vanish, and the metric perturbation is given by

$$A = B = 0, \quad \mathcal{R} = H_L = \lambda. \quad (4.1)$$

From this we obtain

$$\mathcal{A} = -\left(\frac{\mathcal{R}}{H}\right)' = -\frac{3}{2}(1 + w)\lambda. \quad (4.2)$$

Equation (3.7) determines  $\Phi$  from this as

$$\Phi = C\frac{H}{a} + \frac{3}{2}\lambda\frac{H}{a} \int_{t_0} (1 + w)a(t)dt. \quad (4.3)$$

If there exists a dynamical vector field describing matter, this equation and Eq.(3.5) determines  $Z$  as

$$Z = -\frac{2}{3(1 + w)}\mathcal{A} = \lambda. \quad (4.4)$$

On the other hand, if the matter is described only by scalar fields,  $v - B$  is written as  $v - B = k\chi$  where  $\chi$  is a combination of the scalar field perturbations  $\chi_I$  which does not contain a negative power of  $k$ . Since the matter perturbation vanishes in the present case, this term should vanish, which implies that  $Z = \lambda - aH\chi = \lambda$ . Hence the condition (3.5) is satisfied.

Thus we find that the perturbation equations have always two solution for which the  $k \rightarrow 0$  limits of  $\mathcal{A}$ ,  $\Phi$  and  $Z$  are given by Eqs.(4.2)-(4.4). Since the matter is not perturbed,  $\Gamma = 0$  for them. Hence they represent adiabatic modes in the  $k \rightarrow 0$  limit. Clearly the solution proportional to  $C$  is a decaying mode, while that proportional to  $\lambda$  is a growing mode because the Bardeen parameter  $Z$  is a non-vanishing constant. Note that these solutions are universal in the sense that they are valid for any matter contents and interactions.

This universal solutions cover most of the exact solutions in the  $k \rightarrow 0$  limit found so far, and the general argument developed here explains why such exact solutions were found for large variety of matter content. For example, for the case in which matter consists of a single-component perfect fluid, there exist two adiabatic modes. The above solutions just give their  $k \rightarrow 0$  limit, and in terms of the standard variables  $\Delta = \delta\rho/\rho + 3(1 + w)aH(v - B)/k$  and  $V$  they are expressed as

$$\Delta = \frac{2k^2}{3a^2H^2}\Phi = \frac{2k^2}{3Ha^3} \left[ C + \frac{3}{2}\lambda \int_{t_0} (1 + w)adt \right], \quad (4.5)$$

$$V = \frac{k}{aH}(\Phi - Z) = \frac{k}{a^2} \left[ C - \lambda \left( a(t_0)/H(t_0) + \int_{t_0} adt \right) \right], \quad (4.6)$$

which recover the well-known exact solutions[2].



Another example is the case in which matter consists of a multi-component scalar field  $\phi = (\phi_I)$ . In terms of the gauge-invariant variable for the scalar field perturbation defined by

$$X_I = \delta\phi_I - \frac{\dot{\phi}_I}{H}\mathcal{R}, \quad (4.7)$$

the above solution gives

$$X_I = -\frac{\dot{\phi}_I}{H}\lambda, \quad (4.8)$$

and

$$\Phi = C\frac{H}{a} + \lambda\frac{\kappa^2}{2}\frac{H}{a}\int\frac{a\dot{\phi}^2}{H^2}dt. \quad (4.9)$$

These give the extension of the well-known exact solution in the  $k \rightarrow 0$  limit for the single component scalar field case[5, 6, 9].

## §5 Scalar Field Systems

In this section we examine how the condition (3.5) is expressed explicitly for a multi-component scalar field system.

The Lagrangian density for a multi-component scalar  $\phi = (\phi_I)$  is generally expressed as

$$\mathcal{L} = -\sqrt{-g} \left[ \frac{1}{2}g^{\mu\nu}\partial_\mu\phi \cdot \partial_\nu\phi + U(\phi) \right], \quad (5.1)$$

where  $U(\phi)$  is a potential. For a homogeneous background the energy density  $\rho$  and the pressure  $p$  are expressed as

$$\rho = \frac{1}{2}\dot{\phi}^2 + U(\phi), \quad p = \frac{1}{2}\dot{\phi}^2 - U(\phi), \quad (5.2)$$

where  $\dot{\phi}^2 = \dot{\phi} \cdot \dot{\phi}$ . The background equation of motion and the Einstein equations are given by

$$\ddot{\phi} + 3H\dot{\phi} + DU = 0, \quad (5.3)$$

where  $DU = (\partial U/\partial\phi_I)$ , and Eq.(2.20).

Since the energy-momentum tensor for the Lagrangian density (5.1) is given by

$$T_\nu^\mu = \nabla^\mu\phi \cdot \partial_\nu\phi - \frac{1}{2}\delta_\nu^\mu \left( \nabla^\lambda\phi \cdot \nabla_\lambda\phi + 2U \right), \quad (5.4)$$

the perturbation variables for the energy-momentum tensor are expressed as

$$\delta\rho = -A\dot{\phi}^2 + \dot{\phi}\dot{\delta\phi} + DU \cdot \delta\phi, \quad (5.5)$$

$$(\rho + p)(v - B) = \frac{k}{a}\dot{\phi} \cdot \delta\phi, \quad (5.6)$$

$$\delta p = -A\dot{\phi}^2 + \dot{\phi}\dot{\delta\phi} - DU \cdot \delta\phi. \quad (5.7)$$

Hence  $\Delta_g$  and  $Z$  are expressed in terms of the gauge-invariant variable  $X$  for the scalar field perturbation defined by Eq.(4.7) as

$$\rho\Delta_g = -\mathcal{A}\dot{\phi}^2 + \dot{\phi} \cdot \dot{X} + DU \cdot X, \quad (5.8)$$

$$Z = -H \frac{\dot{\phi} \cdot X}{\dot{\phi}^2}. \quad (5.9)$$

Inserting this expression for  $\Delta_g$  into Eq.(2.25), we obtain

$$2U\mathcal{A} + \dot{\phi} \cdot \dot{X} + DU \cdot X = 2\frac{1}{\kappa^2} \frac{k^2}{a^2} \Phi. \quad (5.10)$$

From this, the expression for  $Z$  and the background field equation, it follows that

$$\mathcal{A} + \frac{3}{2}(1+w)Z = -\frac{H}{2U}W\left(\frac{\dot{\phi}}{H}, X\right) + \frac{1}{\kappa^2 U} \frac{k^2}{a^2} \Phi, \quad (5.11)$$

where

$$W(X_1, X_2) := X_1 \cdot \dot{X}_2 - \dot{X}_1 \cdot X_2. \quad (5.12)$$

Hence the condition (3.5) is expressed as

$$\lim_{k \rightarrow 0} W\left(\frac{\dot{\phi}}{H}, X\right) = 0. \quad (5.13)$$

Now let us see what kind of restriction this condition gives for exactly homogeneous perturbations. Let us denote the gauge-invariant variable for an exactly homogeneous perturbation  $\delta\phi$  of the scalar field as  $\mathcal{X} = \delta\phi - \mathcal{R}\dot{\phi}/H$ , where

$$\mathcal{R} = H_L = \delta a/a. \quad (5.14)$$

Then from the perturbation of Eq.(5.3) and Eq.(2.20), we obtain

$$\ddot{\mathcal{X}} + 3H\dot{\mathcal{X}} + D^2U(\mathcal{X}) - \dot{\phi}\dot{\mathcal{A}} + 2DU\mathcal{A} = 0, \quad (5.15)$$

$$2U\mathcal{A} + \dot{\phi} \cdot \dot{\mathcal{X}} + DU \cdot \mathcal{X} = 0, \quad (5.16)$$

where  $D^2U(\mathcal{X}) = (\mathcal{X}_J \partial^2 U / \partial \phi_I \partial \phi_J)$ . Here note that  $\mathcal{X} = \dot{\phi}/H$  and  $\mathcal{A} = 3\dot{\phi}^2/(3\rho) = 3(1+w)/2$  is a solution to this equation, which corresponds to the adiabatic growing mode obtained in the previous section.

Eliminating  $\mathcal{A}$  from these equations we obtain the following second-order differential equation for  $\mathcal{X}$ :

$$L(\mathcal{X}) = -\frac{H^2}{U} \left(\frac{\dot{\phi}}{H}\right) \cdot W\left(\frac{\dot{\phi}}{H}, \mathcal{X}\right), \quad (5.17)$$

where

$$L(\mathcal{X}) := \ddot{\mathcal{X}} + 3H\dot{\mathcal{X}} + \left[ D^2U - \frac{\kappa^2}{a^3} \left( \frac{a^3}{H} \dot{\phi} \circ \dot{\phi} \right) \cdot \right] \mathcal{X}. \quad (5.18)$$

From this it follows that  $W = W(\dot{\phi}/H, \mathcal{X})$  follows the differential equation

$$\dot{W} - \left( 3wH + \frac{1}{U} DU \cdot \dot{\phi} \right) W = 0. \quad (5.19)$$

This equation shows that if  $W$  vanishes at an initial time  $t = t_0$ , it vanishes at any time. Hence Eq.(5.13) reduces to the condition on the initial value of solutions to Eq.(5.17). Therefore, taking into account the fact that  $\mathcal{X} = \dot{\phi}/H$  is one of such solutions, we find that for a  $N$ -component scalar field system  $(2N - 1)$  independent solutions to Eq.(5.17) correspond to the  $k \rightarrow 0$  limit of solutions to the perturbation equation with  $k \neq 0$ .

Here note that we have already obtained two universal solutions in the  $k \rightarrow 0$  limit in the previous section. Among them, the growing mode, coincides with one of the solutions obtained from  $\mathcal{X}$ . On the other hand the decaying mode is not contained in the latter because  $\mathcal{X} = 0$  ( $\delta a/a = 0$ ) for it. Hence we have obtained  $2N$  independent solutions in the  $k \rightarrow 0$  limit. Since the number of the dynamical degrees of freedom of the  $N$ -component scalar field system is  $2N$ , these exhaust all the solutions to the perturbation equations in the  $k \rightarrow 0$  limit. However, if one wants to determine the  $k$ -dependence of the solutions around  $k = 0$  beyond the  $k \rightarrow 0$  limit by solving the perturbation equations iteratively with respect to  $k$ , one needs all the solutions to the perturbation equation for  $X$  in the  $k \rightarrow 0$  limit to find the Green function. Fortunately we can obtain the remaining one independent solution for  $X$  explicitly from a solution for  $\mathcal{X}$  with  $W \neq 0$  in the following way<sup>3</sup>.

First note that the perturbation equation for  $X$  with  $k \neq 0$  is expressed in terms of the operator  $L$  defined in Eq.(5.18) as

$$L(X) + \frac{k^2}{a^2} X = 0. \quad (5.20)$$

From this we see that in the  $k \rightarrow 0$  limit  $X$  satisfies the equation  $L(X) = 0$ , which confirms the above argument.

On the other hand from Eq.(5.19) it follows that

$$\frac{a^3 H^2}{U} W \left( \frac{\dot{\phi}}{H}, \mathcal{X} \right) = \text{const.} \quad (5.21)$$

From this find that

$$L(f\dot{\phi}/H) = \frac{\dot{\phi}}{a^3 H} (a^3 \dot{f})' + 2\dot{f} \left( \frac{\dot{\phi}}{H} \right) \quad (5.22)$$

coincides with the right-hand side of Eq.(5.17) if  $f$  satisfies

$$a^3 \dot{f} = -\frac{a^3 H^2}{2U} W \left( \frac{\dot{\phi}}{H}, \mathcal{X} \right). \quad (5.23)$$

This implies that

$$X = \mathcal{X} + \frac{\dot{\phi}}{H} \int dt \frac{H^2}{2U} W \left( \frac{\dot{\phi}}{H}, \mathcal{X} \right) \quad (5.24)$$

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<sup>3</sup>This point is first suggested out by Sasaki and Tanaka from the consideration on the role of the  $k \rightarrow 0$  limit of  $H_T$  in the discrepancy between the equations for  $X$  and  $\mathcal{X}$  (private communication)

satisfies  $L(X) = 0$  for any solution  $\mathcal{X}$  to Eq.(5.17). Since the second term on the right-hand side of this equation vanishes if  $W(\dot{\phi}/H, \mathcal{X}) = 0$ , and  $W(\dot{\phi}/H, X)$  does not vanish if  $W(\dot{\phi}/H, \mathcal{X}) \neq 0$  from

$$W\left(\frac{\dot{\phi}}{H}, X\right) = \frac{\rho}{U} W\left(\frac{\dot{\phi}}{H}, \mathcal{X}\right), \quad (5.25)$$

$X$  given by Eq.(5.24) exhausts all the solutions to the perturbation equation for  $X$  in the  $k \rightarrow 0$  limit.

## §6 Perfect Fluid Systems

In this section we apply the argument in §3 to a multi-component perfect fluid system as an example of non-trivial systems in which matter variables contain a dynamical vector field.

The equations of motion of a perturbed multi-component system are given by

$$\tilde{\nabla}_\nu \tilde{T}_{I\mu}^\nu = \tilde{Q}_{I\mu} \equiv \tilde{Q}_I \tilde{u}_\mu + \tilde{f}_{I\mu}, \quad (6.1)$$

where  $\tilde{Q}_{I\mu}$  represents the energy-momentum transfer term for the component  $I$ ,  $\tilde{u}^\mu$  is the 4-velocity of the whole matter system, and  $\tilde{Q}_I := -\tilde{u}^\mu \tilde{Q}_{I\mu}$ . Because of the conservation of the total energy-momentum,  $\tilde{Q}_{I\mu}$  satisfies

$$\sum_I \tilde{Q}_{I\mu} = 0. \quad (6.2)$$

For the scalar perturbation, the energy-momentum tensor and the energy-momentum transfer vector of each individual component are expressed as

$$\tilde{T}_{I0}^0 = -(\rho_I + \delta\rho_I Y), \quad (6.3)$$

$$\tilde{T}_{Ij}^0 = a(\rho_I + p_I)(v_I - B)Y_j, \quad (6.4)$$

$$\tilde{T}_{Ik}^j = (p_I \delta_k^j + \delta p_I \delta_k^j + \Pi_I Y_k^j), \quad (6.5)$$

$$\tilde{Q}_{I0} = -[Q_I + (Q_I A + \delta Q_I)Y], \quad (6.6)$$

$$\tilde{Q}_{Ij} = a[Q_I(v - B) + F_{cI}]Y_j, \quad (6.7)$$

where  $\rho_I$ ,  $p_I$  and  $Q_I$  are the background values of the energy density, the pressure and the energy transfer of the component  $I$ , respectively. In terms of these quantities the background part of Eq.(6.1) is written as

$$\dot{\rho}_I = -3h_I(1 - q_I)H, \quad (6.8)$$

where

$$h_I := \rho_I + p_I, \quad q_I := Q_I/(3Hh_I). \quad (6.9)$$

As in §2, we construct from the density perturbation  $\delta\rho_I$ , the velocity perturbation  $v_I$ , the isotropic stress perturbation  $\delta p_I$  and the energy transfer perturbation  $\delta Q_I$  the following

four gauge-invariant combinations:

$$\rho_I \Delta_{gI} := \delta \rho_I + 3(\rho_I + p_I) \mathcal{R}, \quad (6.10)$$

$$V_I := v_I - \frac{a}{k} \dot{H}_T, \quad (6.11)$$

$$p_I \Gamma_I := \delta p_I - c_I^2 \delta \rho_I, \quad (6.12)$$

$$Q_I E_{gI} := \delta Q_I - \frac{\dot{Q}_I}{H} \mathcal{R} \quad (6.13)$$

where  $c_I^2 = \dot{p}_I / \dot{\rho}_I$ . The anisotropic stress perturbation  $\Pi_I$  and the momentum transfer perturbation  $F_{cI}$  of the component  $I$  are gauge-invariant by themselves.

From the relation  $\tilde{T}_{\mu\nu} = \sum_I \tilde{T}_{I\mu\nu}$  the variables for the whole system are expressed in terms of those for each component as

$$\rho = \sum_I \rho_I, \quad p = \sum_I p_I, \quad h = \sum_I h_I, \quad (6.14)$$

$$\rho \Delta_g = \sum_I \rho_I \Delta_{gI}, \quad hV = \sum_I h_I V_I, \quad p\Gamma = \sum_I p_I \Gamma_I + p\Gamma_{rel}, \quad \Pi = \sum_I \Pi_I, \quad (6.15)$$

where  $p\Gamma_{rel} = \sum_I (c_I^2 - c_s^2) \rho_I \Delta_{gI}$ . Further Eq.(6.2) gives the constraints

$$\sum_I Q_I = 0, \quad \sum_I Q_I E_{gI} = 0, \quad \sum_I F_{cI} = 0. \quad (6.16)$$

In terms of these gauge-invariant variables the perturbed equations of motion Eq.(6.1) are written as

$$\begin{aligned} \tilde{\nabla}_\nu \tilde{T}_{I0}^\nu &= \tilde{Q}_{I0} : (\rho_I \Delta_{gI})' + 3H \rho_I \Delta_{gI} + \frac{k^2}{a^2 H} h_I (\Phi - Z_I) + 3H (p_I \Gamma_I + c_I^2 \rho_I \Delta_{gI}) \\ &= Q_I \mathcal{A} + Q_I E_{gI}, \end{aligned} \quad (6.17)$$

$$\begin{aligned} \nabla_\nu T_{Ii}^\nu &= Q_{Ii} : k \left[ \left( \frac{h_I Z_I}{H} \right)' + 3h_I Z_I + h_I \mathcal{A} + p_I \Gamma_I + c_I^2 \rho_I \Delta_{gI} - \frac{2}{3} \Pi_I \right] \\ &= -a F_{cI} + k \frac{Q_I}{H} Z, \end{aligned} \quad (6.18)$$

where

$$Z_I := \Phi - \frac{aH}{k} V_I = \mathcal{R} - \frac{aH}{k} (v_I - B). \quad (6.19)$$

In deriving these equations from the equations of motion Eq.(6.1), we have not multiplied any power of  $k$ . Hence putting  $k = 0$  in these equations yields the equations for the exactly homogeneous perturbations.

Now let us examine the relation of the  $k \rightarrow 0$  limit of these equations and the corresponding equations for the exactly homogeneous perturbations with  $k = 0$ . First note that  $\tilde{p}_I$  and  $\tilde{Q}_I$  depend on the metric and the matter variables through their scalar combinations. In particular the 4-velocities appear in the form  $\tilde{u}_I^\mu \tilde{u}_{J\mu}$  and/or the scalar combinations of  $\tilde{u}_I^\mu$  and its covariant derivatives. In the linear perturbation the terms proportional to  $v_I$  produced from them all come with positive powers of  $k$ . This implies that  $\Gamma_I$  and  $E_{gI}$  are

written as linear combinations of  $\mathcal{A}$ ,  $k^2\Phi$ ,  $\Delta_{gI}$ ,  $k^2Z$  and  $kV_{IJ} = k(v_I - v_J)$  with coefficients which do not contain negative powers of  $k$ . Hence the  $k \rightarrow 0$  limit of Eq.(6.17) and the corresponding  $k = 0$  equation coincide with each other under the condition  $k^2\Phi \rightarrow 0$ .

On the other hand, Eq.(6.18), which vanishes identically for  $k = 0$  due to the vector origin of  $F_{cI}$ , gives a non-trivial equation in the  $k \rightarrow 0$  limit. If we require the condition

$$\Pi_I \rightarrow 0, \quad (6.20)$$

it is written as

$$\left(\frac{h_I Z_I}{H}\right)' + 3h_I Z_I + h_I \mathcal{A} + p_I \Gamma_I + c_I^2 \rho_I \Delta_{gI} = \frac{Q_I}{H} Z + \lim_{k \rightarrow 0} F_{cI}/k. \quad (6.21)$$

Since  $Z$  is related to  $\mathcal{A}$  by Eq.(3.5), this equation can be regarded as one determining the  $k \rightarrow 0$  limit of  $Z_I$ , or  $\lim_{k \rightarrow 0} (v_I - B)/k$ , which has no relation to the quantities describing homogeneous perturbations, from a solution for  $\mathcal{A}$  and  $\Delta_{gI}$  to the perturbation equations with  $k = 0$  (recall the argument on  $\Gamma_I$ ). This is in accordance with the general argument in §3.

In general this gives a set of coupled equations for  $Z_I$ . However, if the condition

$$F_{cI}/k \rightarrow 0 \quad (6.22)$$

is satisfied, it can be explicitly integrated to give

$$h_I Z_I = \frac{H}{a^3} \left[ C_I - \int_{t_0} dt a^3 (h_I \mathcal{A} + p_I \Gamma_I + c_I^2 \rho_I \Delta_{gI} - 3h_I q_I Z) \right], \quad (6.23)$$

where  $C_I$  is an integration constant and  $t_0$  is an initial time. In particular, for the universal adiabatic modes given in §4, this gives

$$h_I Z_I = \frac{H}{a^3} \left[ C_I + \lambda \left\{ \frac{a^3}{H} h_I - \left( \frac{a^3}{H} h_I \right)_0 \right\} \right], \quad (6.24)$$

where the subscript 0 denotes the value at  $t = t_0$ . Here note that Eq.(6.22) is a rather strong condition in realistic situations where  $F_{cI}$  is usually proportional to the relative velocities  $V_{IJ}$ .

As discussed in §3, in order for this  $k \rightarrow 0$  limit solution to correspond to a  $k = 0$  solution, the consistency conditions (3.3) and (3.4) should be satisfied. Under the requirement (6.22), these conditions are simply reduced to the asymptotic condition

$$k C_I(k) \rightarrow 0. \quad (6.25)$$

Further,  $Z_I$  is also restricted by the  $k \rightarrow 0$  limit equation (3.5). Since  $Z$  is expressed as

$$\begin{aligned} hZ &= \sum_I h_I Z_I \\ &= \frac{H}{a^3} \left[ \sum_I C_I + \int_{t_0} dt a^3 (-h\mathcal{A} - p\Gamma - c_s^2 \rho \Delta_g) \right] \\ &= \frac{H}{a^3} \left[ \sum_I C_I - \frac{2}{3} \left\{ \frac{a^3}{H} \rho \mathcal{A} - \left( \frac{a^3}{H} \rho \mathcal{A} \right)_0 \right\} \right], \end{aligned} \quad (6.26)$$

from Eq.(2.25) and Eq.(2.30), this condition is expressed as

$$\sum_I C_I + \frac{2}{3} \left( \frac{a^3}{H} \rho \mathcal{A} \right)_0 = 0. \quad (6.27)$$

This is just the equation (3.5) at the initial time  $t = t_0$ .

For a  $N$ -component perfect fluid system, the perturbation of the exactly homogeneous solutions generates  $N$ -independent solutions for  $\Delta_{gI}$  and  $\mathcal{A}$ , which satisfy the  $k \rightarrow 0$  limit of Eq.(6.17) and the Einstein equations. Each of these solutions in turn determines  $Z_I$  through Eq.(6.23) with  $N - 1$  independent integration constants satisfying the condition (6.27). By this procedure we obtain  $2N - 1$  independent solutions to the whole perturbation equations in the  $k \rightarrow 0$  limit. Thus, by adding the universal adiabatic decaying mode corresponding to the trivial homogeneous perturbation, we can obtain all the  $2N$  independent  $k \rightarrow 0$ -limit solutions for this system from the exactly homogeneous solutions.

## §7 Summary and Discussions

In this paper we have investigated the relation between the  $k \rightarrow 0$  limit of solutions to the perturbation equations with  $k \neq 0$  and the exactly homogeneous perturbations ( $k = 0$ ) for quite general multi-component systems on a spatially flat Robertson-Walker universe. The main result is summarized as follows.

First for the case in which the fundamental variables describing cosmic matter contain a dynamical vector field, for any solution to the perturbation equations with  $k = 0$  there exists a solution to the perturbation equations with  $k \neq 0$  which coincides with the former in the  $k \rightarrow 0$  limit. On the other hand if cosmic matter consists only of scalar fields, such a correspondence holds if and only if the exact homogeneous perturbation satisfies an additional constraint corresponding to the momentum constraint which is expressed by a linear equation on the initial condition for the homogeneous perturbation. This implies that for the  $N$ -component scalar field system one can obtain the  $k \rightarrow 0$  limit of  $2N - 1$  independent solutions to the perturbations equations if one knows all the solutions to the homogeneous background equations.

We have also shown that from trivial exactly homogeneous perturbations one can always construct two adiabatic solutions to the perturbation equations in the  $k \rightarrow 0$  limit which can be explicitly expressed in terms of integrals of background quantities, irrespective of the content and the interactions of cosmic matter. One of them corresponding to the vanishing homogeneous perturbation represents a decaying mode for which the Bardeen parameter vanishes. The other corresponds to a simple constant scaling of the scale factor and gives a growing mode whose Bardeen parameter is a non-vanishing constant. This constant scaling is not a gauge transformation, but an extra symmetry of the background equations which holds only for spatially homogeneous Robertson-Walker universes. Thus this extra symmetry is the hidden reason why exact solutions in the  $k \rightarrow 0$  limit have been so far found for various systems in spite of the non-trivial structure of their perturbation equations.

For the multi-component scalar field system these universal adiabatic modes and the solutions obtained from the non-trivial exact homogeneous perturbations exhaust all the

solutions to the perturbation equations in the  $k \rightarrow 0$  limit. Hence the evolution of perturbations on superhorizon scales can be determined only from the knowledge on the homogeneous solutions to the background equations in the lowest-order approximation. However, if one wants to know the higher-order correction in  $k$ , one must solve the perturbation equations  $k \neq 0$  iteratively [6]. In this procedure one needs the  $k \rightarrow 0$  limit of all the solutions to the perturbation equations for the gauge-invariant variable describing the perturbation of the scalar fields. As we have shown, this information can be also obtained from the exactly homogeneous solutions to the perturbation equations. Thus the behavior of superhorizon perturbations of this system can be determined with any accuracy from the knowledge on the homogeneous solutions.

This argument can be directly extended to the scalar field system coupled with radiation, for which all the  $k \rightarrow 0$  limit of the solutions to the perturbation equations are directly determined from the homogeneous background solutions. This correspondence may provide a very powerful method to analyze the evolution of perturbations during reheating in the inflationary models because the structure of the equations for the homogeneous scalar fields decaying to radiation is much simpler than that for the perturbation equations with energy transfer terms.

## Acknowledgments

The authors would like to thank K. Nakao, Y. Nambu, M. Sasaki, T. Tanaka, A. Taruya and K. Tomita for valuable discussions and comments, without which this paper would not have emerged. T. H. would like to thank Prof. H. Sato for continuous encouragements. H. K. is supported by the Grant-In-Aid of for Scientific Research (C) of the Ministry of Education, Science, Sports and Culture in Japan(05640340).

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